Eine Kleine Fourier Musik

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1. DFT of a pc set

We state some basic definitions and facts; the context is a tempered chromatic universe, whose model is $\mathbb{Z}_c$.

1.1. Syllabus. Following David Lewin's short note at the end of his very first paper([**4**]),

DEFINITION 1. *The DFT of a subset $A \subset \mathbb{Z}_c$ is the Discrete Fourier Transform of its characteristic function:*

$$t \mapsto \mathcal{F}_A(t) = \mathcal{F}(1_A)(t) = \sum_{k \in A} e^{-2i\pi kt/c}$$

If A is a multiset, we keep the same definition (each k occuring with its multiplicity)

We recall Inverse Fourier Transform, Parseval's theorem, and the link with convolution product. This provides a

1.2. Connection with interval function. The interval function between two subsets is a convolution product between some characteristic functions. Hence

COROLLARY 1 (The Original Application by Lewin). *A subset A is determined by its intervalic relation with another, except when its DFT vanishes (Lewin's special cases).*

Here notice already the importance of roots of a DFT. The interval vector of a lone pc subset is:

$$IC_A = 1_A \star 1_{-A} \text{ (convolution product), hence } \quad \mathcal{F}(IC_A) = \mathcal{F}_A \times \mathcal{F}_{-A} = |F_A|^2$$

- From there it is easy to get the usual results on the complement $\mathbb{Z}_c \setminus A$, e.g. different forms of hexachordal theorem.
- This means that two pc-sets A, B are $Z-$related (they share the same interval vector) $\iff |\mathcal{F}_A| = |\mathcal{F}_B|$.

2. Maximal values of DFT

2.1. Some special cases. For a regular polygon, all intervals are multiples of one of them. For instance a diminished seventh has minor thirds and multiples, and nothing else. This periodicity is of course striking on the DFT (that's the whole point of Fourier transform !): in the case of the diminished seventh, the DFT is nil except for multiples of 4.

PROPOSITION 1. *Subset A with cardinality d is a regular polygon $\iff \mathcal{F}_A(d) = \mathcal{F}_A(0) = d$.*

For a cluster (consecutive pcs on the chromatic circle), we get another nice characterisation:

PROPOSITION 2. *The d elements of A are consecutive $\iff$ $|\mathcal{F}_A(1)|$ is maximal amongst all d-elements subsets.*

Meaning that all the $e^{2ik\pi/c}$ 'pull in the same direction'.

2.2. Quinn's alternate definition. In [**5**] it is noted that Maximally Even Sets (in $\mathbb{Z}_{12}$ at least) share an interesting property. We have proved this can be taken as an actual definition of ME Sets:

DEFINITION 2 (À la Lewin). *Subset $A \in \mathbb{Z}_c$ is a ME set with d elements $\iff$ $|\mathcal{F}_A(d)|$ is maximal among all subsets of cardinality d.*

It follows immediately that translates and inverses of a ME set are still ME sets. Almost as obvious is that the notion is invariant under complementation. The point of course is to prove that this notion of ME set is indeed the usual one. Sketchily, if A is a ME set in the above sense:

(1) Maximising $|\mathcal{F}_A(d)|$ is maximising $|\mathcal{F}_{dA}(1)|$;
(2) hence dA is a cluster, i.e. (1,2,...d) up to translation;
(3) hence $A = (f, 2f, 3f, \dots df)$ wherein $f = d^{-1}$.

2.3. The periodical case $(c, d) = \gcd(c, d) > 1$. It is roughly the same thing but

(1) Now $\widehat{A} = dA$ has to be construed as a multiset (the map $x \mapsto dx$ being no longer one-to-one), and
(2) We maximize $\mathcal{F}_{\widehat{A}}(1)$, not amongst multisets in $\mathbb{Z}_c$ but in $d\mathbb{Z}_c$, which is the subgroup generated by (c, d).
(3) This turns out to be the same thing as building a ME set in a quotient of $\mathbb{Z}_c$ (see [**2**]), which completes the proof.

2.4. Generated scales. Also characterised by a maximal value of DFT, as they are affine transform of a ME set with same cardinality. So:

PROPOSITION 3. *A is a generated scale with* $\operatorname{Card} A = d \iff \|\mathcal{F}_A\|_\infty = \max_{t=1\dots c-1} |\mathcal{F}_A(t)|$ *is maximal (among all subsets of cardinality d).*

This maximum can be easily computed from a ME set with card d.

3. 0 is a minimum

3.1. Roots of $\mathcal{F}_A$. Here we have some Galois theory.

THEOREM 1. *If $\mathcal{F}_A(t) = 0$ then all elements of $(\mathbb{Z}_c, +)$ (the additive group) with the same order as t are roots too. They are the $k \times t$ where k is any integer coprime with c.*

Basically this means that the cyclotomic polynomials are irreducible.

COROLLARY 2. *The roots of $\mathcal{F}_A$ are invariant under affine action, i.e. replacing A with $kA + b$ (k coprime with c).*

3.2. Tiling questions. The structure of the zero-set $Z_A = \{k \in \mathbb{Z}_c, \mathcal{F}_A(k) = 0\}$ of $\mathcal{F}_A$ is very closely related to the question of tiling (whether chord, or rhythm).

DEFINITION 3. *A tiles in $\mathbb{Z}_c$ whenever there exists a subset B such that $A \oplus B = \mathbb{Z}_c$, or equivalently $\forall t$ $\mathcal{F}_A \times \mathcal{F}_B(t) = 1 + e^{-2i\pi t/c} + \dots e^{-2i\pi(c-1)t/c}$ or again $Z_A \cup Z_B = \{1, 2, \dots c-1\}$ and* $\operatorname{Card} A \times \operatorname{Card} B = c$.

The **conditions of Coven-Meyerowitz** for tiling ([**3**]) distinguish the roots of order p^α from the other ones ([**1**]).

Also particular tilings may be distinguished immediately from their DFT: for instance Vuza canons, as $k-$periodicity means DFT is nil except on a subgroup of $\mathbb{Z}_c$.

3.3. Spectral Sets. In 1974 Fuglede introduced a famous conjecture, concisely expressed as "tiling $\iff$ spectral". It remains undecided in dimensions 1 and 2.

DEFINITION 4. *A subset A of $\mathbb{Z}_c$ is spectral $\iff$ there exists $\Lambda \subset \mathbb{Z}_c$ (with same cardinality) such that $\Lambda - \Lambda$ is a subset of the roots Z_A of $\mathcal{F}_A$ (0 excepted).*

4. Perspectives

4.1. Classifications. As seen for Z-relation, the **absolute value** of the DFT is equivalent to the interval vector; More generally, there is a hierarchy of more or less refined classifications according to values of DFT:

- Values of $|\mathcal{F}_A|$: characterises the interval vector of A. This is the (mathematical) Z-relation.
- $|\mathcal{F}_A|$ reaches a maximal value (among $d-$subsets) somewhere: A is generated (an arithmetical progression)
- $|\mathcal{F}_A(\text{Card}\,A)|$ is maximum (in the same sense): A is a ME set.
- $|\mathcal{F}_A|$ keeps around the mean value: most, or all, intervals are present.

  CONJECTURE. *Set $A \in \mathbb{Z}_c$ with d elements is all-interval $\iff$ $\mathcal{F}_A$ stays as close as possible (amongst all d-elements subsets) to mean value $\sqrt{\dfrac{(c-1)d}{d-1}}$.*

- List of roots of $\mathcal{F}_A$: an invariant under affine transform, it could be the key to tiling condition (AND spectral condition).

4.2. Multisets. Why and how we must look at them.
$A - A$ is computed first as a multiset, and it is convenient to keep it that way: as people involved in musical software will know quite well, computer wise one gets some intervals several times, which is precisely the interest of the interval vector. So a very important multiset appears here as a difference set.
Multisets also crop up with affine maps which are not one to one (see above for periodical ME sets).
A good way to compute with them is to use their DFT, as it turns the awkward convolution product $1_A \star 1_B$ (useful though both for tiling and interval questions) into ordinary product.
If time allows, this talk will also present a matrix representation (of multisets) in a $c-$dimensional Cartan algebra, wherein diagonalisation is just another way of expressing the DFT.

4.3. Getting closer to Fuglede?... To sum it up in a nutshell, the spectral condition can be symbolically expressed as

$$\mathcal{F}(A - A)(\Lambda - \Lambda) = 0$$

Closer scrutiny of multisets and/or intervalic relations might be the key to the elucidation of the 1 dimensional case of the conjecture.

References

[1] Amiot, E., *Why Rhythmic Canons are Interesting*, in: E. Lluis-Puebla, G. Mazzola et T. Noll (eds.), *Perspectives of Mathematical and Computer-Aided Music Theory, EpOs*, 190–209, Universität Osnabrück, 2004.
[2] Clough, J., Douthett, J., *Maximally even sets*, in: *Journal of Music Theory* (1991), 35:93173.
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[4] Lewin, D., *Re: Intervalic Relations between two collections of notes*, in: *Journal of Music Theory*, (1959) 3:298-301.
[5] Quinn, I., *A Unified Theory of Chord Quality in Equal Temperaments* , (2004) Ph.D. dissertation: Eastman School of music.