Isographies of pitch-class sets and set classes

Tuukka Ilomäki
tuukka.ilomaki@iki.fi

Introduction

One of the major differences between tonal and atonal music is the significantly larger number of available pitch-class sets in the latter. The categorization of these pitch-class sets and the analysis and classification of their relations stand as a major strand in the 20th century music theory. The equivalence relation induced by the group of transpositions and inversions is the *de facto* standard classification of pitch-class sets. Music theorists, however, have recently suggested alternative approaches, for instance, relations based on similarity or voice leading.

Klumpenhouwer networks or K-nets present yet another approach, this one based on transformation theory. K-nets provide a way to create relations beyond the limits of set classes. Several authors have contributed to this research: Henry Klumpenhouwer, David Lewin, Philip Lambert, and many others. Furthermore, the idiomatic theory of George Perle provides means to tackle the K-nets (**?**; **?**).

A K-net contains some specified contents as vertices and transformations as arrows between the vertices. Typically, the vertices are pitch classes and the arrows are transpositions and inversions. The basic idea of this paper is very simple: we examine what the isography of K-nets (with distinct pitch classes as vertices) reveals us about the pitch-class sets involved.

Figure 1 depicts a typical pair of positively isographic K-nets in the middle. Two K-nets with pitch classes $\{0, 1, 6\}$ and $\{0, 5, 8\}$ are isographic via $\langle \mathrm{T7} \rangle$. Taking this isography of K-nets as our starting point, we can focus either on the transformations or on the contents of the vertices. In other words, we omit or ignore some information. First, to the left we omit the contents of the vertices, leaving a K-graph. David Lewin (**?**) has pursued this direction in his studies of the relations between transformation networks. Second, to the right we omit the transformations, leaving the underlying pitch-class sets.This direction, and the distinction between moving left or right in this way, has been less explored. Philip Lambert (**?**) has studied trichordal K-class families with fixed configurations of transpositions, inversions and pitch-class locations. In this paper, however, we focus on the underlying pitch-class sets, with fixed configurations of transpositions and inversions, but allowing any configuration of pitch-class locations. Our aim is to explore what kind of pitch-class sets can be arranged into strongly or positively isographic K-nets using maximally even configurations 2+1 for trichords, 2+2 for tetrachords, etc.[1] We will move from isographic pitch-class sets to isographic set classes and group set classes by shared isographic potential.

Isography of pitch-class sets and set classes

The term "isography" has been used to denote an automorphism of transformation networks (or, more specifically, certain type of automorphism, since the automorphisms involving the M-operation are

[1]Without loss of generality we can confine ourselves to dealing only with positive isography since, in the context of set classes, negative isography of pitch-class sets A and B implies positive isography of pitch-class sets A and T_nIB.

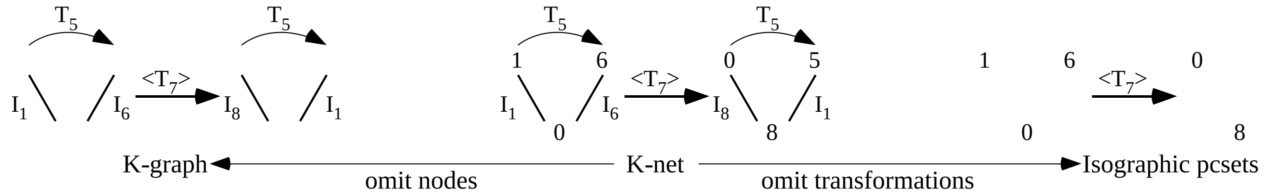

FIGURE 1: From K-nets to K-graphs and isography of pitch-class sets.

usually excluded). The change of context from transformation networks to pitch-class sets requires us to redefine isography. We retain the term even if the two are algebraically rather different domains.

DEFINITION 1 Two pitch-class sets are (strongly) isographic if and only if the pitch classes can be arranged into K-nets that are (strongly) isographic.

By definition, isography of K-nets implies isography of the pertinent pitch-class sets. Two K-nets with pitch classes of two isographic pitch-class sets are not necessarily isographic, however, since the isography of K-nets depends on both the pitch classes and their configuration. For instance, K-nets with pitch classes $\{0, 1, 6\}$ and $\{0, 5, 8\}$ are isographic only if the transpositional relation in the K-nets is between pitch classes 1 and 6 and pitch classes 0 and 5, respectively (as in figure 1).

We derive the isography of set classes from the isography of pitch-class sets. Theorems 1 and 2 guarantee that the isography of set classes is well defined. Hence, the isography of set classes does not depend on the selection of pitch-class sets representating the set classes.

DEFINITION 2 Two set classes are (strongly) isographic if and only if there are two pitch-class sets belonging to the set classes that are (strongly) isographic.

THEOREM 1 If A and B are (strongly) isographic pitch-class sets, then T_nA and T_nB are also (strongly) isographic pitch-class sets and so are T_nIA and T_nIB.

THEOREM 2 Let A and B be (strongly) isographic pitch-class sets that belong to set classes $SC(A)$ and $SC(B)$, respectively. If A' is an arbitrary pitch-class sets belonging to $SC(A)$ then there exists a pitch-class set B' belonging to $SC(B)$ such that A' and B' are (strongly) isographic.

Pitch-class sets related by transposition are isographic but not necessarily strongly isographic. Pitch-class sets related by inversion are negatively isographic but not necessarily positively isographic. For instance, transpositionally related pitch-class sets $\{0, 1, 2\}$ and $\{1, 2, 3\}$ are positively isographic but not strongly isographic; inversionally related pitch-class sets $\{0, 1, 2, 3, 4, 6\}$ and $\{0, 2, 3, 4, 5, 6\}$ are negatively isographic but not positively isographic.

Tonality and whole-tone scale proportion

Pitch-class sets can be classified using two concepts: *tonality* and *whole-tone scale proportion*. Originally, George Perle defines the tonality of tetrachords as the sum of the four pitch classes modulo 4. In addition, sums 1 and 3 are considered equivalent.[2]

The following definitions generalize tonality in all cardinalities. For the sake of coherency, the definition must be stated in terms of the greatest common divisor of the cardinality n and 12. Hence, tonality is a useful concept only in cardinalities n such that $\gcd(n, 12) > 1$.

[2]We use sums of pitch classes as a notational shorthand; we do not posit a group structure on the set of pitch classes.

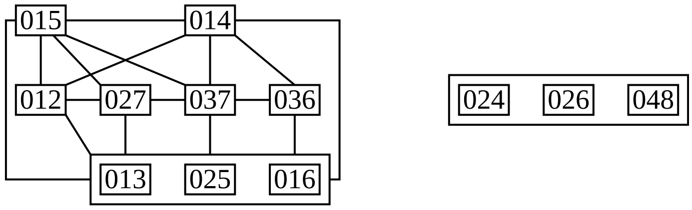

FIGURE 2: Strong isographies of trichords.

DEFINITION 3 The tonalities of cardinality n are the equivalence classes that the automorphisms of the group $\mathbb{Z}_{\gcd(n,12)}$ induce.

DEFINITION 4 The tonality of a pitch-class set of cardinality n is the equivalence class of the sum of its pitch classes $\mathrm{mod}\ n$.

For example, the tonalities of cardinality 4 are $\{0\}$, $\{1,3\}$, and $\{2\}$, and the tonalities of cardinality 6 are $\{0\}$, $\{1,5\}$, $\{2,4\}$, and $\{3\}$. The tonality of pitch-class set $\{0,1,2,3,4,5\}$ is $\{3\}$.

THEOREM 3 Pitch-class sets A, T_n, and T_nIA have the same tonality.

Since all pitch-class sets that belong to the same set class have the same tonality, we can extend the concept of tonality to the set classes.

DEFINITION 5 The tonality of a set class is the tonality of any of its constituent pitch-class sets.

The proportion of odd and even intervals in a pitch-class set depends only on how the pitch classes divide between the two whole-tone scales. Isography of set classes involves the odd and even intervals; hence, the two whole-tone scales play a role as one decisive factor behind isography. This proportion is invariant in transpositions and inversions (and the M-operation); hence, the number of odd and even intervals in pitch-class sets belonging to the same set class is constant (as evinced by the fact that pitch-class sets in the same set class have identical interval-class vectors).

Relations of set classes

The relations that isography defines on the set of pitch-class sets and the set of set classes are reflexive and symmetric but not transitive. Hence, we do not obtain an equivalence relation. Set classes are grouped into categories with two layers, however, since strong isography exists only within each category and positive isography (without strong isography) exists only between certain categories.

In all cardinalities, the whole-tone scale proportion defines a boundary condition on isography. In the even cardinalities tonality imposes an additional boundary condition. In cardinalities 2 and 10, however, tonality and whole-tone scale proportion coincide. Furthermore, in cardinality 3 set classes with the same whole-tone scale proportion are either strongly isographic or not isographic. Hence, all non-strong isographies are between set classes with different whole-tone scale proportion. Correspondingly, in cardinalities 4 and 6 set classes of the same tonality are either strongly isographic or not isographic. Hence, all non-strong isographies are between set classes of different tonalities. These conditions do not apply to the complement cardinalities 8 and 9.

Figure 2 depicts the strong isographies of the trichords. The set classes are divided into those that contain pitch classes from only one whole-tone scale and those that contain pitch classes from both whole-tone scales. Figure 3 depicts the strong isographies of the tetrachords. Whole-tone scale proportion and tonality divide the set classes into five distinct categories. Hence, the relations defined by

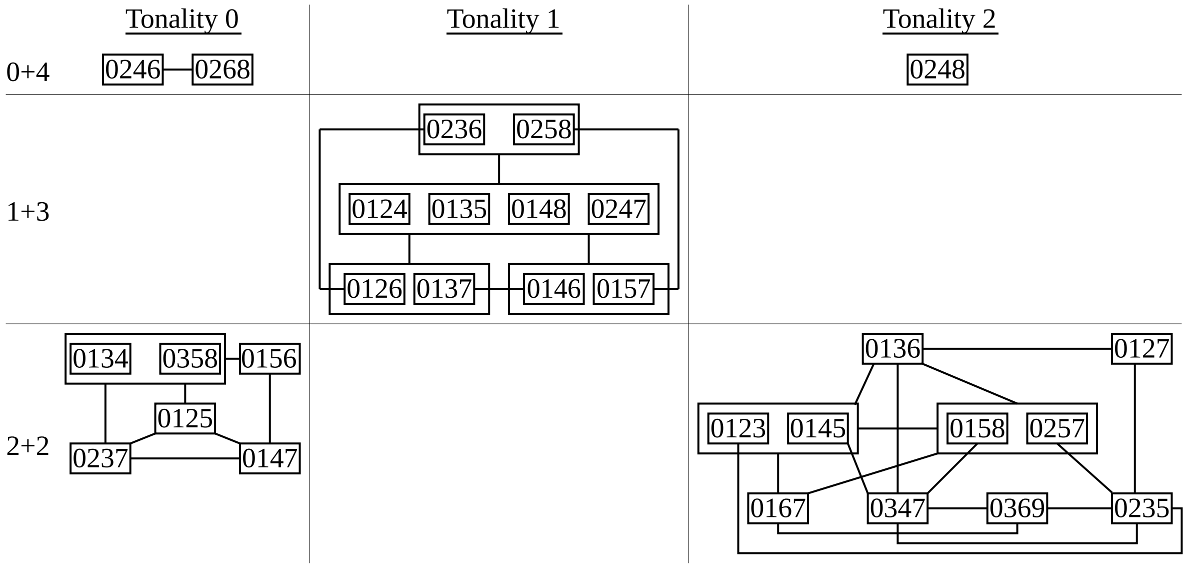

FIGURE 3: Strong isographies of tetrachords.

strong isography are subsets of the equivalence relations that the whole-tone scale proportion induces on the set of trichords and the whole-tone scale proportion and tonality induce on the set of tetrachords.

Theorems 4 and 5 involve complementary pitch-class sets and set classes of the complementary cardinality. Their proofs follow immediately from the observation that $\gcd(n, 12) = \gcd(12 - n, 12)$ for all $1 \leq n \leq 12$ and that there exists an automorphism of group $\mathbb{Z}_{\gcd(n,12)}$ that maps $k \pmod{\gcd(n, 12)}$ into $66 - k \pmod{\gcd(n, 12)}$.

THEOREM 4 The tonalities in cardinalities n and $12 - n$ are identical.

THEOREM 5 Complementary pitch-class sets have the same tonality.

Theorems 4 and 5 show that complementary set classes group together correspondingly. Octachords, for instance, divide into five categories and three tonalities like tetrachords. The complementary set classes have the same tonality. The larger cardinalities are more flexible, however: there are more ways to arrange 8 pitch classes than 4 pitch classes into the vertices of a K-net. Consequently, the larger cardinalities permit more strong isographies than the smaller ones. Even if the corresponding set classes in complementary cardinalities divide into corresponding tonalities, we cannot deduce the isography of two set classes from the (strong) isography of the corresponding set classes of the complement cardinality. For instance, the all-interval tetrachord set classes 4-Z15[0146] and 4-Z29[0137] are strongly isographic, but the corresponding complementary set classes 8-Z15[01234689] and 8-Z29[02346789] are not even isographic. Similar examples can be found between pentachords and septachords (5-1[01234] and 5-Z12[01356] versus 7-1[0123456] and 7-Z12[0123479]) and trichords and nonachords (3-1[012] and 3-11[037] versus 9-1[012345678] and 9-11[01235679A]).

The Z-related set classes provide a convenient test on whether we could base our theory on the interval-class content of the set classes. The answer is negative. We must take into account the specific pattern of pitch classes in the set classes. For example, set class 6-Z4[012456] is strongly isographic with 7 hexachords, whereas its Z-related set class 6-Z37[012348] is strongly isographic with 4 hexachords. Indeed, these two Z-related set classes are not even isographic.